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From duration analysis to GARCH models – An approach to systematization of quantitative methods in risk measurement[1]

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Abstract: The development of scientific research has led to the very dynamic growth of methods in the area of financial risk management. This refers particularly to risk measures in which quantitative methods are applied. The paper provides a discussion on a systematization of different risk measures proposed in scientific literature and used in practice. There are four criteria proposed in the paper. The first is the concept of risk applied by distinguishing negative and neutral concept. The second criterion is the character of the risk variable, either discrete or continuous. The third criterion makes the distinction between high frequency, low severity events, corresponding to standard (normal) type of risk, and low frequency, high severity events, corresponding to extreme risk. Finally the fourth criterion distinguishes between the risk variable expressed in monetary values and risk variable expressed in time units. Using these criteria the most common groups of risk measures are discussed. The final part of the paper gives a synthetic discussion on model risk which is a risk resulting from the erratic model used in a real world. In the paper three main sources of model risk are presented and the methods to evaluate model risk are given.

Keywords: risk measures, extreme risk, model risk, volatility measures, sensitivity measures.

JEL codes: C02, C58, D53, G32.

Introduction

Risk management is one of the most dynamically developed areas in economic sciences. No doubt, one of the main driving forces for this development has been the practical challenge resulting from increasing financial risk. Risk management is a process in which key role is played by risk measurement.

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There have been many approaches proposed to measure risk, both by theoreticians and practitioners. The aim of the paper is to systematize some risk measures with regard to several criteria as well as provide some related comments. The most important scientific milestones in the area of risk measurement are:
- Duration [Macaulay 1938], proposed as a measure of interest rate risk for debt instruments;
- Portfolio theory [Markowitz 1952 and 1959] in which variance as a risk measure was proposed to analyze market risk (mostly stock price risk);
- Separation theorem [Tobin 1958] being an extension of portfolio theory;
- Use of discriminant analysis in credit risk measurement [Altman 1968];
- Option pricing theory [Black and Scholes 1973; Merton 1994], providing a framework to analyze option as risk hedging tool;
- Family of GARCH models [Engle 1982; Bollerslev 1986], an adaptive method to calculate market risk.

The risk measures mentioned were proposed many years ago, the last being (GARCH model) about thirty years ago. In the framework discussed below we concentrate on these measures. Since that time there has not been new significant methodological development. Many researchers made some surveys of these measures, including Brachinger and Weber [1997] as well as Pedersen and Satchell [1998]. On the contrary there were many applications of existing measures.

The most commonly measured types of financial risk are:
- Market risk, which is risk resulting from the possible changes of prices on financial and commodity market (stock price risk, interest rate risk, exchange rate risk, commodity price risk, real estate price risk);
- Credit risk, which is risk resulting from the possibility that the other party of the contract will not make a contractual payment;
- Operational risk, which is risk resulting from failures of internal systems, human errors or external events.

The paper is structured as follows. The first section outlines criteria for the systemization of financial risk measures. The next sections deal with particular cases of risk measures, namely: risk measures based on continuous distribution (Section 2), sensitivity measures (Section 3), measures of extreme risk (Section 4), risk measures for risk variable expressed in time units (Sections 5 and 6). The last section discusses model risk. The paper is closed with brief conclusions.

1. Financial risk measures – criteria for systematics

There are very many different criteria that can be used to systematize risk measures. From the point of view of quantitative tools that can be applied in risk measurement we propose the following four criteria:

– Negative (one-sided) or neutral (two-sided) concept of risk;
– Continuous or discrete risk variable;
– Risk considered in monetary units or in time units;
– Normal risk or extreme risk.

Negative or neutral concept of risk

In the negative concept risk is understood as a threat and therefore it can be described by the possibility of falling below some expected or desired level. Such a concept is the only one used in insurance, also it is found in the other areas of finance where negative consequences are of crucial importance. Typically negative concept of risk is applied with regard to operational risk and to credit risk (with the exception of credit risk of tradable debt instruments such as bonds).

In the neutral concept risk is understood as a threat on the one hand and as an opportunity on the other. Typically a neutral concept of risk is applied with regard to market risk.

Continuous or discrete risk variable

This is a very important distinction since it has impact on the quantitative tools used in risk measurement. The most common continuous risk variables are: price, return, loss. Such a variable is measured on a ratio scale. These variables are treated as continuous for practical reasons despite the fact that the actual number of values they can have is countable – they are measured with some degree of accuracy, e.g. one cent. However treating them as continuous variables brings the benefits of the possible use of a plethora of quantitative tools including those based on continuous distributions. Discrete risk variables can usually have several values. Very often they correspond to the occurrence of some event. The simplest case is the binary risk variable which takes value 1 when the risk event (such as the default of the company) occurs and value 0 when risk event does not occur.

Risk considered in time and monetary units

Despite the fact that this distinction is usually not discussed in literature we decided to take it into consideration since it bears some importance in understanding risk. When risk is considered in „monetary" units, then the time period is given and risk is modelled through a „monetary" variable such as value or return. A typical example is return, being the variable analyzed in a given period. When risk is considered in time units then monetary value is given and risk is modelled through a time variable. A typical example is the time needed to achieve some value of investment.

Normal risk or extreme risk

The distinction between these two types of risk comes from the fact that risk events can occur with different frequencies and have an impact of different severity (reflected by the size of the possible loss). „Normal" risk results from the events which have a high frequency and low severity. On the other hand extreme risk results from the events which have a low frequency and high severity.

We propose four criteria for the systematization of risk measures. Each one has two possible categories. Therefore if these criteria are combined we can theoretically have 16 classes of risk measures. However some classes should be eliminated for quite obvious reasons. First of all, extreme risk is analyzed only in a negative sense since it is defined through the occurrence and size of losses. Secondly, when the discrete risk variable is considered it is also analyzed in a negative sense, since it is related to the occurrence of the risk event. Thirdly, when risk is considered in time units it is usually taken as a continuous variable and the tools suitable for continuous variables are used.

This finally leads us to 6 classes of risk measures corresponding to the following cases:

1. Neutral concept, continuous risk variable, „normal" risk, measured in „monetary" units.
2. Negative concept, continuous risk variable, „normal" risk, measured in „monetary" units.
3. Negative concept, continuous risk variable, „normal" risk, measured in time units.
4. Negative concept, continuous risk variable, extreme risk, measured in „monetary" units.
5. Negative concept, discrete risk variable, „normal" risk, measured in „monetary" units.
6. Negative concept, discrete risk variable, „normal" risk, measured in time units.

In the next sections we will discuss typical risk measures and refer to the cases mentioned.

2. Risk measures based on continuous distribution

This is probably the oldest and the most well-known group of risk measures. The most common use of these measures is in two cases:

1. Neutral concept, continuous risk variable, „normal" risk, measured in „monetary" units.
2. Negative concept, continuous risk variable, „normal" risk, measured in „monetary" units.

Here risk is being captured by the statistical distribution of the risk variable. We can distinguish three types of measures:

- Volatility measures;
- Quantiles of distribution;
- Values of distribution function.

Volatility measures have their origin in the seminal paper by Markowitz [1952] who proposed the use of the variance of returns as a measure of risk. Later practitioners started to use the standard deviation of risk variable as a risk measure. Of course other statistical measures can be used such as: mean deviation or interquartile range.

Volatility measures can be put in a more general framework based on the so-called L_p-norm. Here the volatility measure is defined as:,

$$\sigma = \left[\int_{-\infty}^{\infty} \left(R - L(R)\right)^p f(R) dR \right]^{1/p},$$

where:

$L(R)$ – the reference value,
R – risk variable,
$f(R)$ – density function.

It is worth mentioning that this general framework contains the following cases:

1. For $p = 1$:
   $L(R)$ – equal to the median of the risk variable;
   σ – denotes mean deviation from the median of the risk variable.
2. For $p = 2$:
   $L(R)$ – equal to the expected value of the risk variable;
   σ – denotes standard deviation of risk variable.
3. For $p=\infty$:
   $L(R)$ – equal to the average of maximal and minimal value of the risk variable;
   σ – denotes half of the difference between the maximal and minimal value of the risk variable.

The presented presented are used when neutral concept of risk is considered. However most of them can be relatively easily adapted to the situation when the negative concept of risk is considered. It can be seen that volatility measures are constructed in such a way that deviations, both positive and negative, from a reference value (mean, median, etc.) are taken into account. Therefore one can modify these measures by taking only negative deviations from the reference value. Thus we get such measures such as semi variance, semi standard deviation, etc.

There is a general approach used as a modification of volatility measures when the negative concept is used. It is the Lower Partial Moment, defined as:

$$\sigma = \begin{cases} \left(\int_{-\infty}^{RT} |R - RT|^p f(R) dR \right)^{1/p} & \text{if} \quad 0 < p < \infty, \\ \int_{-\infty}^{RT} f(R) dR & \text{if} \quad p = 0, \end{cases}$$

where

RT – target value of risk variable.

There are at least three cases of lower partial moment:

1. For $p = 0$ the lower partial moment is the value of the distribution function at the point equal to the target value.
2. For $p = 1$, if the target value is assumed to be the expected value of the risk variable then, for symmetrical distribution, the lower partial moment is equal to half of the mean deviation of the risk variable.
3. For $p = 2$, if the target value is assumed to be the expected value of the risk variable then the lower partial moment is equal to the semi standard deviation of the risk variable (which is a square root of its semi variance).

The other two groups of measures are used only when one considers the negative concept of risk.

The quantile of the distribution of the risk variable can be interpreted in the following way: The probability that the risk variable will fall below the quantile is very low so the end user can conclude that it is very unlikely that the realization of the risk variable will be below the quantile. Here quantile can be treated as "safety level". The concept of safety level was introduced by Roy [1952]. The value of the distribution function of risk variable can be interpreted in the following way: It gives the probability that the risk variable will not exceed a certain level.

3. Sensitivity measures

These measures are derived from the function linking the risk variable to risk factors. The most general description of this function is given as:

$$R = g(X_1, X_2, \ldots, X_m),$$

where:

R – risk variable,

X_i – risk factors,

g – risk function.

A sensitivity measure (denoted as beta) is defined with regard to each risk factor as:

$$\beta_i = \frac{\partial g(X_1, \dots, X_m)}{\partial X_i}.$$

A sensitivity measure is formally defined as the partial derivative of the risk function with regard to a particular risk factor. This is a "what if" type of a measure since it results in the change of the risk variable given that the risk factor changes by some small unit whilst the other factors remain unchanged. In literature very many sensitivity measures have been proposed, for example: duration, beta or Greek coefficients related to option pricing.

Although sensitivity measures were originally proposed using the neutral concept of risk they can also be used when one adopts them for the negative concept. Then only unfavourable changes of risk factors are taken into account. Therefore these measures can be used in two ways:

1. Neutral concept, continuous risk variable, „normal" risk, measured in „monetary" units.
2. Negative concept, continuous risk variable, „normal" risk, measured in „monetary" units.

Of course in the second case one considers only these changes of risk factors that lead to negative consequences.

4. Measures of extreme risk

Measures of extreme risk are used in the following case: negative concept, continuous risk variable, extreme risk, measured in „monetary" units.

Some of the measures presented based on statistical distribution of risk variables can be applied also in this case. Particularly it refers to:

– Quantiles of distribution – here one should take quantile as corresponding to the very low probability;
– Value of the distribution function – here one should take the „extreme" value of the risk variable.

There is however a group of risk measures that were inherently designed to be used in extreme risk measurement. The most commonly used are based on the so-called Conditional Tail Distribution or Conditional Excess Distribution. It is given as a distribution function of so-called excess:

$$F_u(r) = P(R - u \le r \mid R > u).$$

The distribution function of conditional tail distribution is defined as the probability that the risk variable (loss) will exceed some threshold value (denoted by u) by the amount of r or lower, given that it exceeds this threshold value. In a common sense interpretation: "if things go wrong, how bad can they actually be".

The most common extreme risk measure is the Expected Shortfall (or Expected Tail Loss) given as the expected value of the conditional excess distribution by the following formula:

$$ES = E(R|R > u).$$

The interpretation of Expected Shortfall is simple: it shows what is – on average – the value of the risk variable if this variable exceeds some threshold value. Clearly, the higher the ES, the higher extreme risk.

5. Risk variable measured in time units – continuous variable

These measures correspond to the following case: negative concept, continuous risk variable, „normal" risk, measured in time units. Here the most appropriate risk measure is based on the distribution of the time variable, being a continuous variable. It is the value of distribution function, given as:

$$P(T < \tau) = \int_0^\tau f(T)dT,$$

where:

T – time variable,

$f(T)$ – the density function of the distribution of the random variable T,

τ – fixed time unit.

The interpretation of this measure is as follows: the probability that the time to reach some value is shorter than a fixed time unit.

6. Risk measurement for discrete variable

When risk variable is discrete we think of some risk event. This corresponds to two cases:

1. Negative concept, discrete risk variable, „normal" risk, measured in „monetary" units.
2. Negative concept, discrete risk variable, „normal" risk, measured in time units.

The first case is the situation where one looks for the probability that the risk event happens at a considered time, thus "how likely is it that the risk event happens in next period". The second case is when one tries to determine the period in which the risk event happens, thus "how long one will wait until risk event happens".

The common framework used here is based on the statement then when the risk event occurs, one faces loss. This loss can be shown in the following way:

$$L = R \cdot LGD,$$

where:

L – loss variable,

R – risk variable, corresponding to the risk event,

LGD – variable, defined as a loss given the risk event.

The decomposition given by the above formula refers to the situation where an entity is facing risk because of the occurrence of the risk event. This occurrence is reflected by a binary risk variable, R. This variable takes on two possible values: 0 – when the risk event does not occur and 1 – when the risk event occurs. The second component of this decomposition is the so-called *Loss Given Risk Event* variable which is continuous. Clearly, if the risk event occurs the loss is equal to LGD, if the risk event does not occur the loss is equal to 0. The most common practical example is when the risk event refers to the default of the counterparty and this default results in the lack of payment. Then the LGD is simply the *Loss Given Default*.

If we consider the discrete risk variable as being a binary variable then the risk measure is simply the probability of the risk event, defined as:

$$PD = P(R = 1),$$

and

$$E(R) = PD.$$

So the probability of the risk event is also the expected value of the risk variable.

This relationship leads to another – derived for the loss:

$$E(L) = E(R)E(LGD|R) = PD \cdot LGD.$$

In practice it can happen that the risk variable is not binary but has more categories. For example, we can categorize the risk variable by identifying several risk categories, e.g., "very low risk", "low risk", "medium risk", "high risk", "very high risk". Such a situation is faced, for example, when a rating is being conducted by a rating agency.

Suppose that there are m categories of risk (ordered and disjointed). They are denoted by the numbers from 1 to m, where 1 denotes the highest risk. Since here risk is understood in its negative sense the natural extension of the measure proposed for the binary variable – the probability of the risk event – can be proposed. This is the probability that the risk category will be below some stated level. It is given by the following formula:

$$PD = P(R < a) = P(R = 1) + P(R = 2) + \ldots + P(R = a - 1),$$

where:

a – stated level, being the lowest accepted risk category.

Of course the higher the value of PD, the higher the risk. Similarly to the case of binary variable here we also have the relation to loss variable, given as:

$$E(L) = \sum_{i=1}^{m} P(R = i) E(L \mid R = i).$$

So the expected loss is the weighted average of losses conditional on each risk category whilst the weights are the probabilities that the risk variable belongs to this category.

7. Model risk

Models are scientific concepts used to describe the real world. In some disciplines, particularly natural sciences, many models have a deterministic nature and they match the real world perfectly. In other disciplines, particularly social sciences, most models are only an approximation to the real world and they are very often of a stochastic nature. It is because human behaviour cannot be predicted and captured by deterministic structures. This means that the application of a model by the end user can lead to the results that are not expected. It is called **model risk**.

The notion "model risk" can be defined in a general and simplified way: **Model risk is a risk resulting from an erratic model used in the real world.** The issue of model risk is especially relevant for models derived within financial science, for example, financial market models. In this paper model risk is analyzed from the point of view of the investor who makes financial investments of different types and relies on different models to support investment decisions. Models used within the area of the financial market are directed at the valuation of financial instruments and to decision making process in the financial market. On one hand, these models try to describe real world; on the other, they support the decision making process of market participants (investors).

A very good common sense explanation of model risk existing in financial models is given by Robert Merton. He stated (see Merton [1994]), that: „At times we can lose sight of the ultimate purpose of the models when their mathematics becomes too interesting. The mathematics of financial models can be applied precisely, but the models are not all precise in their application to the complex real world. Their accuracy as a useful approximation to that world varies significantly across time and place. The models should be applied in practice only tentatively, with careful assessment of their limitations in each application".

There are at least three distinct sources of uncertainty resulting in model risk.

1. Uncertainty as to the structure of the model.
   This refers to the situation in which some important factors (variables) are omitted; there is the wrong functional form of relationship between the variables in the model, etc.
2. Uncertainty as to the parameters of the model.
   Even if there is a correct structure of the model some parameters of the model usually need to be estimated. Model risk occurs since there are different estimation methods, possibilities of using different data sets, etc.
3. Uncertainty as to the application of the model.
   This refers to the situation where model that was correctly constructed and estimated is not a good description of the actual case.

Using financial models without a clear understanding of the uncertainty lying behind them may lead to consequences in the financial market, including severe losses. The first significant case of losses resulted from model risk occurred during collapse of the hedge fund Long Term Capital Management in 1998. Managers of this fund used rather sophisticated financial models and performed extremely well in the period of stable markets. However their models were not properly adjusted for the case of extreme shocks which happened during the East Asian crisis in 1997 and the Russian crisis in 1998. High leverage taken by this fund led to huge losses and collapse, however there was a bailout of this fund.

Financial models are not necessarily transferrable in time or in space. This is because of the way the market is functioning can be different in time and space. One of the main errors made by designers and users of models is an attempt to get the best fit of these models to historical data. It is however clear that past data may be not relevant for future and that is the main interest of financial investors.

In my opinion models used by investors on the financial market should satisfy two constraints. First of all, they have to be robust to the changes of market conditions, secondly they have to be transparent to the end user. Satisfying the first constraint can diminish risk in the application of the model given the market changes. Satisfying the second constraint means decreasing the risk that the appropriate model is used in the wrong way by the end user who does not understand the model. In my opinion good practice is to supplement financial model used in practice with an analysis of the risk of this model. One can propose different approaches to analyze model risk.

If there is uncertainty as to the structure of the model and uncertainty as to the application of the model then qualitative judgment should be used. This judgment will contain the analysis of the assumptions lying behind the model as well as the analysis of the particular market in which the model is to be applied. In the case of the uncertainty as to the parameters of the model, one can adopt the quantitative approach. Here we propose the application of standard quantitative measures of market risk. There are two groups of such measures:

1. Distribution based measures.
   Here one analyzes the distribution of the estimate of the parameter. The larger dispersion of this distribution, the higher is risk resulting from the estimation of this parameter.
2. Sensitivity measures.
   Here one analyzes the sensitivity of the output of the model to the changes of the value of the respective parameter. From the mathematical point of view the sensitivity measure is the derivative of the output of the model with regard to the parameter. The higher the sensitivity, the higher is the risk resulting from the estimation of this parameter.

There are also other tools that can be used in the analysis of model risk occurring in financial models. Practitioners apply the following two approaches:

a) Back testing, being the verification of the model using past data – if the model does not perform well on the past data this might be an indication of uncertainty as to the application of this model in future;
b) Stress testing, being the verification of the model using data generated subject to extreme shocks – this shows how sensitive the model is to such shocks; usually the occurrence of instability is not a good recommendation for the model.

Conclusions

The main conclusions derived from the considerations of this paper refer to the systematization of risk measures. The proposed criteria enabled a listing the main groups of risk measures. This can be used as a hint for possible use by practitioners.

One of the most important is the distinction between normal (standard) risk and extreme risk. This results from the consequences led by extreme risk. On the other hand, the estimation of extreme risk measures is a more difficult task.

From the point of view of methodology used in risk measurement the type of risk variable is a key factor. A continuous risk variable is the most appropriate for market risk measurement. A discrete risk variable is the most often used for credit risk measurement. The crucial practical conclusion is related to the notion of model risk. This is risk resulting from an erratic model used in a real world. It is advised that the model risk of risk measures is evaluated to avoid misuse of a particular measure.

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