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Emil PANEK
Poznań University of Economics

Optimal paths in the endogenous AK class growth models[1]

Abstract: The AK class models are simple visions of economic dynamics, where production level is determined by quantity of physical capital at economy's disposal and efficiency of its usage in the process of production. One assumes implicitly that technological conditions unambiguously define the level of technical endowment of labor (the ratio of capital to labor) and labor resources are so abundant that they serve any quantity of capital.

In the paper we show that even in such simple aggregated models of economic dynamics, where an endogenous growth mechanism of capital efficiency works (due to investments and innovations by R&D), it is possible to achieve social welfare, measured by production and consumption p.c., which is not attainable on the grounds of the neoclassical growth models.
Keywords: endogenous growth model, human capital, optimal growth path, optimal consumption (production) rate, turnpike effect.
JEL codes: O41.

1. The basic AK model[2]

We assume that there exists only one good which is limited and – depending on use – plays the role of production good (production factor) or consumption good, e.g. corn. The other necessary production factors such as labor, land, water, air etc. are assumed to exist in practically unconstrained quantities. Land does not need

[1] The paper is a part of the Author's larger work "Kapitał ludzki jako czynnik wzrostu gospodarczego. Ujęcie modelowe (na przykładzie zagregowanych modeli klasy AK)", published in the monography: "Nowe trendy w metodologii nauk ekonomicznych i możliwości ich zastosowania w procesie kształcenia ekonomicznego", Publ. UEP, Poznań 2010 (in Polish).

[2] The term "AK class models" usually denotes simplest growth models whose common feature is constant productivity of capital. Research over economic growth grounded on the AK class models was initiated at the beginning of the XX century by Harrod (1939) and W. Leontief, see: Barro & Sala-i-Martin (2003), ch. 4; Gomułka (1990), ch. 15, sect. 15.9; Panek (2003), ch. 8, sect. 8.1–8.3, ch. 9, sect. 9.4; Rebello (1991).

to be fertilized – it is naturally fertile. The economy works in continuous real time. The time variable is denoted by *t* and it is assumed that time runs over an interval $T = [0, t_1] \subseteq R^1_+$, which is called the functioning *horizon* of economy.

At every moment $t \in T$ it is possible to produce $A > 0$ units of good from a unit of good.

The stock of good used in production is said to be *production capital* (or simply: *capital*). The quantity of capital used at moment *t* is denoted by $K(t)$. So at every moment *t* from capital $K(t)$ can be produced

$$Y(t) = AK(t) \tag{1}$$

units of good. Quantity $Y(t)$ (*production*) represents the effects of economic activity, that is to say the product of capital $K(t)$.

We assume that capital $K(t)$, and in consequence its product $Y(t)$, is perfectly divisible, i.e. functions $K(t)$, $Y(t)$ admit any real (non-negative) values and that production capacities are unlimited and the economy is able to produce quantity $Y(t)$ of product from capital $K(t)$ at any moment according to the *linear production function* (1). The coefficient *A* is called *capital productivity*. In the basic version of AK model capital productivity is constant in time because there are no (internal or external) sources of human capital improvement (e.g. in the form of innovative investments).

At every moment $t \in T$ a fraction $s(t) \in [0, 1]$ of product $Y(t)$ is spent on the *(gross) capital investment*

$$I_b(t) = s\,(t)\;Y(t) \tag{2}$$

that enables capital accumulation, and the left fraction $1 - s(t)$ of production is set apart for *consumption*

$$C(t) = (1 - s(t)\;Y(t). \tag{3}$$

A unit of gross investments $I_b(t)$ brings $\sigma > 0$ units of new capital (capital accumulation effect), but on the other hand at every moment *t* capital wears off at a (constant) rate $\mu > 0$ (capital depreciation effect).

The parameter σ is called *investment efficiency rate*, and the parameter μ – *capital depreciation rate*. The difference

$$I_b(t) = \sigma I_b(t) - \mu K(t) \tag{4}$$

is called the *net investment*. The dynamics of capital in the economy in horizon *T*, follows accumulation and depreciation effects and is given by the differential equation

$$\dot{K}(t) = I_n(t)$$

or equivalently (from (1), (2), (4)):

$$\dot{K}(t) = \big(\sigma As(t) - \mu\big) K(t). \tag{5}$$

The coefficient

$$s(t) = \frac{I_b(t)}{Y(t)} \in [0,1]$$

is *gross investment rate at t* (*investments rate* in short), the coefficient

$$\sigma s(t) - \frac{\mu}{A} = \frac{I_n(t)}{Y(t)}$$

is *net investment rate*, and

$$1 - s(t) = \frac{C(t)}{Y(t)} \in [0,1]$$

stands for *consumption rate at t*.

Let

$$K(0) = K_0 > 0 \tag{6}$$

be a fixed initial stock of capital at $t = 0$ and

$$s(t) = s \in [0, 1]$$

at every $t \in T$. The solution of differential equation (5) under (initial) condition (6) and for an integrable function $s{:}T \to [0, 1]$ is every trajectory (growth path) of capital

$$K(t) = K_0 \exp \int_0^t \big(\sigma As(\theta) - \mu\big) d\theta,$$

with corresponding trajectories (growth paths) of product

$$Y(t) = Y_0 \exp \int_0^t \big(\sigma As(\theta) - \mu\big) d\theta$$

($Y_0 = aK_0$), gross investment

$$I_b(t) = As(t) K_0 \exp \int_0^t \big(\sigma As(\theta) - \mu\big) d\theta,$$

net investment

$$I_n(t) = \left(\sigma As(t) - \mu\right) K_0 \exp \int_0^t \left(\sigma As(\theta) - \mu\right) d\theta$$

and consumption

$$C(t) = A\left(1 - s(t)\right) K_0 \exp \int_0^t \left(\sigma As(\theta) - \mu\right) d\theta.$$

The model represents one-dimensional continuous one-dimensional, where investment rate $s(t)$ plays the role of control variable (input), capital $K(t)$ is state variable, and consumption $C(t)$ is output variable. Different values of control variables induce different states (capital) and output variables (consumption).

Our task is to trace out a trajectory which is in a sense better (not worse) than the others.

2. Preferences in the AK-economy with diminishing marginal utility and time discounting

Every economic activity finally leads to satisfaction of consumption needs in a broad sense. Production and investment enable realization of the ultimate goal. Therefore, in the economic theory, and in mathematical economics in particular, a special role of various 'consumption' criteria is emphasized when evaluating growth processes, and one of the most common criterion is maximization of the discounted utility of consumption in the horizon $T = [1, t_1]$:

$$U\left(C(t)\right) \exp(-\rho t),$$

where $u \in C^2\left((0, +\infty) \to R^1\right)$ an instantaneous utility of consumption meeting standard assumptions:

$$\forall C > 0 \left(\frac{du(C)}{dC} > 0 \,\&\, \frac{d^2u(C)}{dC^2} < 0 \right), \tag{7}$$

$$\lim_{C \to 0^+} \frac{du(C)}{dC} = +\infty, \tag{8}$$

and $\rho > 0$ is the time *discount factor.*

The dynamics of consumption in our economy depends on investment rate $s(t)$ and capital $K(t)$, whose dynamics is given by (5) (under initial condition (6)). We assume that control variable $s(t)$ belongs to the class of piecewise differentiable functions with co-domain [0,1], and a finite number of discontinuity points of the first type in the interior of time horizon T, and is right-continuous at every discontinuity point[3]. The class of functions is denoted by $\tilde{C}^0(T \to [0,1])$.

The solution of differential equation (5) corresponding to a given function $s \in \tilde{C}^0(T \to [0,1])$ is a continuous piecewise differentiable function $K(t)$ which satisfies the equation everywhere but at the points of discontinuity of $s(t)$ in T. We say that the solution of (5) is solution of differential equation in *integral sense*.

We are interested in solving the following optimal control problem:
find

$$\max \int_T u\big(C(t)\big)\exp(-\rho t)dt \tag{9}$$

under constraints

$$\begin{gathered} C(t) = A\big(1 - s(t)\big)K(t), \\ \dot{K}(t) = \big(\sigma A s(t) - \mu\big)K(t), \\ s \in \tilde{C}^0\big(T \to [0,1]\big), \\ K(0) = K_0 > 0. \end{gathered} \tag{10}$$

Every pair of functions $s(t)$, $K(t)$ satisfying conditions (10) is said to be a *feasible growth process*. The function $s(t)$ is called a *feasible trajectory* (*growth path*) *of investment rate*, and the function $K(t)$ is called a *feasible trajectory* (*growth path*) *of capital* in the economy. The corresponding, according to (1) – (4), functions $Y(t)$, $I_b(t)$, $I_n(t)$, $C(t)$ are called *feasible trajectories* (*growth paths*) *of product*, *gross investment*, *net investment* and *consumption*, respectively.

The growth process $s^*(t)$, $K^*(t)$ that maximizes functional (9) over the set of all feasible trajectories is called the *optimal process*. The function $s^*(t)$ is called the *optimal trajectory* (*growth path*) *of investment rate*, the function $K^*(t)$ is the *optimal trajectory* (*growth rate*) of *capital,* and the corresponding functions $Y^*(t)$, $I_b^*(t)$, $I_n^*(t)$, $C^*(t)$ are called the *optimal trajectories* (*growth paths*) *of product*, *gross investment*, *net investment* and *consumption*, respectively.

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[3] This is the narrowest class of functions for which the below defined optimal control problems possess solutions, see: Chiang (1992), ch. 7–10; Pontriagin, et al. (1961); Rojtenberg (1978). From a mathematical point of view right-continuity of $s(t)$ (at discontinuity points) is unimportant for the existence of solutions of (5) and is introduced only for clarity of presentation.

To obtain solution of problem (9)-(10) we use Pontriagin's maximum principle[4]. Under assumption of relatively high capital productivity and high efficiency of capital investment in comparison to capital depreciation rate and the discount factor[5],

$$\sigma A > \mu + \rho, \tag{11}$$

the optimal solution is as follows:

There exists a moment $\tau > 0$, such that if horizon T is short ($t_1 \leq \tau$), then the optimal solution is given by the growth paths

$$s^*(t) = 0, \tag{12}$$

$$K^*(t) = K_0 \exp(-\mu t), \tag{13}$$

$$Y^*(t) = Y_0 \exp(-\mu t), \tag{14}$$

$$C^*(t) = C_0 \exp(-\mu t), \tag{15}$$

in T, where $Y_0 = C_0 = AK_0$.

If the horizon T is long enough (it suffices that $t_1 > \frac{1}{\mu+\rho} \ln \frac{\sigma A}{\sigma A - \mu - \rho}$), then there exists a function $\psi(t)$, such that trajectories $s^*(t)$, $K^*(t)$, $C^*(t)$ in $[0, \tau)$ (in the first phase of growth) are solutions of the system of differential equations:

$$\dot{K}(t) = \left(\sigma As(t) - \mu\right) K(t), \tag{16}$$

$$\dot{\psi}(t) = -\left(\sigma As(t) - \mu\right)\psi(t) - A\left(1 - s(t)\right) u'_{C(t)} \exp(-\rho t), \tag{17}$$

$$u'_{C(t)} = \sigma\psi(t)\exp(\rho t), \tag{18}$$

$$C(t) = A\left(1 - s(t)\right) K(t) \tag{19}$$

under (initial) condition (6), where $u'_{C(t)} = \left.\frac{du(C)}{dC}\right|_{C = C(t)}$.

The system should be explicitly solvable if analytical form of the instantaneous utility function $u(C)$ is given. For example if

[4] Some basic information on Pontriagin's maximum principle may be found in positions listed in footnote 2.

[5] Low levels result in the - unacceptable from an economic point of view – optimal solution $s^*(t) \equiv 0$ over T, and that is why we omit this case.

$$u(C) = \ln C, \tag{20}$$

then, by (18), we get

$$u'(C) = C^{-1} = \sigma\psi(t)\exp(\rho t),$$

and after simple transformations we get

$$s^*(t) = 1 - \frac{\exp(-\rho t)}{\sigma A K(t)\psi(t)}. \tag{21}$$

Substituting (18) to (17) results in a simple equation

$$\dot{\psi}(t) = -(\sigma A - \mu)\psi(t),$$

which is satisfied in interval $[0, \tau]$ by function

$$\psi(t) = \psi_0 \exp\{-(\sigma A - \mu)t\}.$$

If we put the function to (21), then we obtain the following formula for the optimal investment rate over interval $[0, \tau)$[6]:

$$s^*(t) = 1 - \frac{\exp\{(\sigma A - \mu - \rho)t\}}{\sigma A\left(\psi_0 K^*(t)\right)} < 1, \tag{22}$$

where there is actually capital $K^*(t)$ on the right-hand-side of the equality, but this inconvenience can be easily excluded by substitution of $C^*(t)$ in place of the investment rate $s^*(t)$ in (22) (and taking into account (19)). Then we have:

$$u'_{C^*(t)} = \frac{1}{C^*(t)} = \sigma\psi(t)\exp(\rho t) = \sigma\psi_0 \exp\{-(\sigma A - \mu - \rho)t\},$$

which allows us to write the optimal trajectory of consumption in interval $[0, \tau)$ as:

$$C-(t) = \frac{1}{\sigma\psi_0}\exp\{(\sigma A - \mu - \rho)t\}.$$

The corresponding optimal capital trajectory $K^*(t)$ in $[0, \tau)$ is obtained by solving the differential equation:

[6] In our problem the optimal investment rate never reaches its upper bound $s^*(t) = 1$, which is a consequence of condition (7), concavity of the instantaneous utility function $u(C)$ and condition (8) (so-called Inada's condition for $C \to 0^+$).

$$\dot{K}(t) = \sigma\big(AK(t) - C^*(t)\big) - \mu K(t)$$

under initial condition (6). The solution obviously satisfies condition $s^*(t) = 1 - \frac{C^*(t)}{Y^*(t)}$, where $Y^*(t) = AK^*(t)$.

Reassuming: If the time span $T = [0, t_1]$ is short, then the optimal solution of problem (9)-(10) is given by (12)-(15) with zero investment rate and negative $(-\mu)$, everywhere on T, growth rates of capital, production and consumption.

When the horizon $T = [0, t_1]$ is long then the optimal growth process has two phases: the first one is investment phase and the second – consumption phase.

In the optimal process the optimal investment rate has the form:

$$s^*(t) = \begin{cases} 1 - \dfrac{\exp\{(\sigma A - \mu - \rho)t\}}{\sigma A \psi_0 K^*(t)} & \text{for} \quad t \in [0, \tau), \\ 0 & \text{for} \quad t \in [\tau, t_1], \end{cases} \tag{23}$$

where the capital $K^*(t)$ at the moment $t = \tau$ in (23) satisfies:

$$K^*(\tau) = \frac{\exp\{(\sigma A - \mu - \rho)\tau\}}{\sigma A \psi_0}. \tag{24}$$

The corresponding optimal consumption trajectory is

$$C^*(t) = \begin{cases} \dfrac{\exp\{(\sigma A - \mu - \rho)t\}}{\sigma \psi_0} & \text{for} \quad t \in [0, \tau), \\ AK^*(t) & \text{for} \quad t \in [\tau, t_1], \end{cases} \tag{25}$$

and the optimal production trajectory is $Y^*(t) = AK^*(t)$ Condition (24) ensures continuity of the optimal investment rate $s^*(t)$ over the horizon T (see Fig. 1), and in the same time – continuity of the optimal investment rate and consumption trajectories (these are smooth everywhere but at τ).

Calculation of the analytical expression of the initial value ψ_0 in (24)-(25) is difficult and we need to employ some numerical procedures, whereas the "switching" moment τ may sometimes be expressed analytically: for example when $\mu = \rho = 0$, one can show

$$\tau = t_1 - \frac{1}{\sigma A}.$$

If the interval of feasible investment rate in problem (9) – (10) is constrained to a subset $[s_0, s_1] \subset (0, 1)$, then the optimal trajectory of investment rate has the form shown on Fig. 2, growth rate of the optimal consumption in the first phase is:

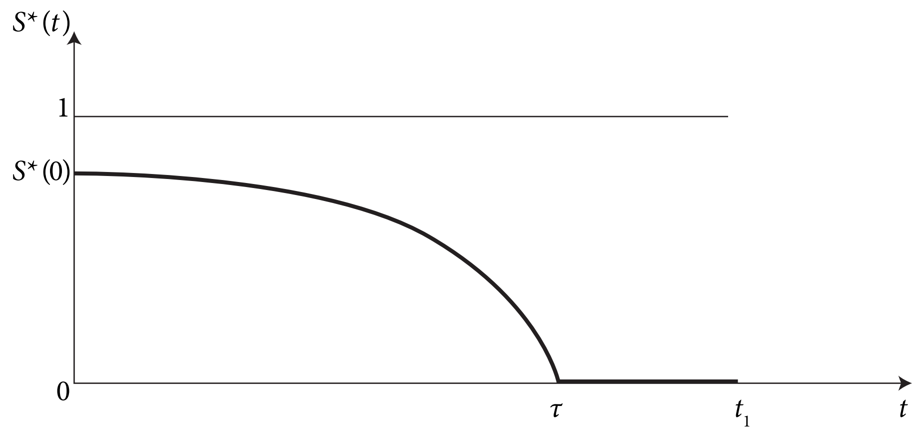

Figure. 1. The optimal investment rate – solution of problem (9)-(10) (long horizon)

$$g_{C^*(t)} = \frac{\dot{C}^*(t)}{C^*(t)} = \sigma A s_1 - \mu - \rho,$$

and this phase's growth rates of capital and production vary:

$$g_{K^*(t)} = \frac{\dot{K}^*(t)}{K^*(t)} = g_{Y^*(t)} = \frac{\dot{Y}^*(t)}{Y^*(t)} \in (\sigma A s_0 - \mu - \rho, \sigma A s_1 - \mu - \rho).$$

In the second phase, capital, production and consumption grow at equal rate:

$$g_{K^*(t)} = g_{C^*(t)} = g_{Y^*(t)} = \sigma A s_0 - \mu.$$

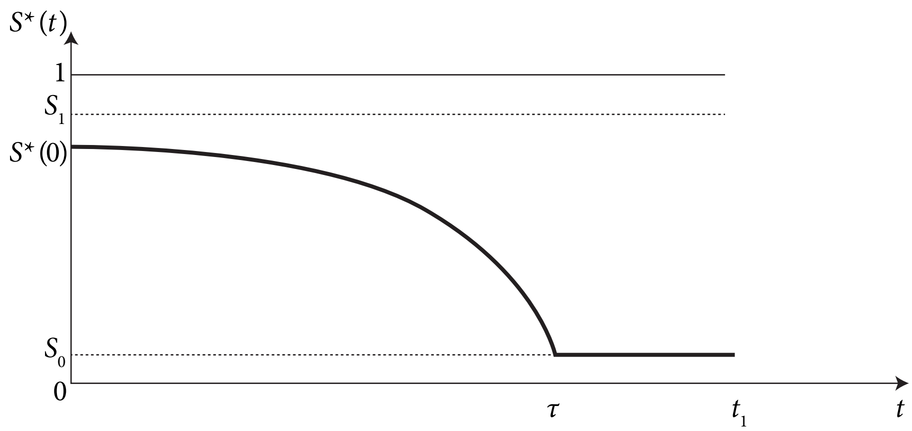

Figure 2. The optimal investment rate in the economy – solution of (9) – (10) under restricted investment rates $[s_0, s_1] \subset (0, 1)$ (long horizon)

As the horizon T gets longer, the first phase gets longer, too. The length of phase two is always limited. In the next section we shall consider a model where T grows unboundedly and we shall observe consequences of the horizon changes for optimal trajectories properties in the economy.

Observe that growth rates of basic variables in the solution of problem (9)-(10) are uniformly bounded (where defined) and independent of the horizon length T. More precisely, for no horizon T the growth rate of capital, production or consumption does not exceed $g_{max} = \sigma A - \mu$.

The bound becomes much stronger after limiting feasible investment rate values to an interval $[s_0, s_1] \subset (0, 1)$.

3. Infinite horizon ($t_1 = +\infty$). The turnpike effect

Under standard assumptions on utility function and parameters we formulated in the previous section we are interested in solving the following problem:

Find

$$\max \int_0^\infty u\big(C(t)\big) exp(-\rho t) dt \tag{26}$$

satisfying (11),

which differs from problem (9)-(10) only in that time horizon T , under which we search for optimal process, is unbounded from above.

The solution of problem (26) is obtained by application of a modified Pontriagin's maximum principle[7]. The solution is given by the optimal trajectories of capital $\overline{K}(t)$, production $\overline{Y}(t)$ and consumption $\overline{C}(t)$ which are defined over time span $T = [0, +\infty)$ as follows:

$$\overline{K}(t) = K_0 \exp\{(\sigma A - \mu - \rho)t\}, \tag{27}$$

$$\overline{Y}(t) = A\overline{K}(t), \tag{28}$$

$$\overline{C}(t) = C_0 \exp\{(\sigma A - \mu - \rho)\, t\}, \tag{29}$$

where $\sigma C_0 = \rho K_0$, and corresponding investment rate

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[7] See Chiang (1992), ch. 9.

$$\overline{s}(t)=\overline{s}=1-\frac{\rho}{\sigma A}=const.\in(0,1). \tag{30}$$

Pair $\overline{s}(t)$, $\overline{K}(t)$ is said to be the *optimal balanced growth process* in the AK-economy with constant capital productivity. Growth paths $\overline{s}(t)$, $\overline{K}(t)$, $\overline{Y}(t)$, $\overline{C}(t)$ are called the *turnpikes* (investment, capital, production and consumption, respectively). There is a simple connection between turnpikes and solutions to (36) with bounded horizon ($t_1 < +\infty$). Namely, when $t_1 \to +\infty$, then the optimal investment rate trajectory $s^*(t)$ (the optimal solution of problem (9)-(10)) is at every moment asymptotically convergent to its turnpike level $\overline{s}=1-\frac{\rho}{\sigma A}$ (so-called pointwise convergence), and the optimal trajectories of capital $K^*(t)$, production $Y^*(t)$ and consumption $C^*(t)$ are at every moment asymptotically convergent to their respective turnpike levels $\overline{K}(t)$, $\overline{Y}(t)$ and $\overline{C}(t)$.

To be more specific, let $s^*_{t_1}(t)$, $K^*_{t_1}(t)$, $Y^*_{t_1}(t)$ and $C^*_{t_1}(t)$ be the optimal trajectories of investment rate, capital, production and consumption, respectively, at moment t in solution of (10)-(11) under a bounded horizon $T = [0, t_1]$, $t_1 < +\infty$), and let $\overline{s}(t)=\overline{s}$, $\overline{K}(t)$, $\overline{Y}(t)$, $\overline{C}(t)$ be turnpikes (of form (27)-(30)). Then for every t, when $t_1 \to +\infty$:

$$\left|s^*_{t_1}(t)-\overline{s}\right|\to 0,$$

$$\left|K^*_{t_1}(t)-\overline{K}(t)\right|\to 0,$$

$$\left|Y^*_{t_1}(t)-\overline{Y}(t)\right|\to 0,$$

$$\left|C^*_{t_1}(t)-\overline{C}(t)\right|\to 0.$$

(See Fig. 3–5)

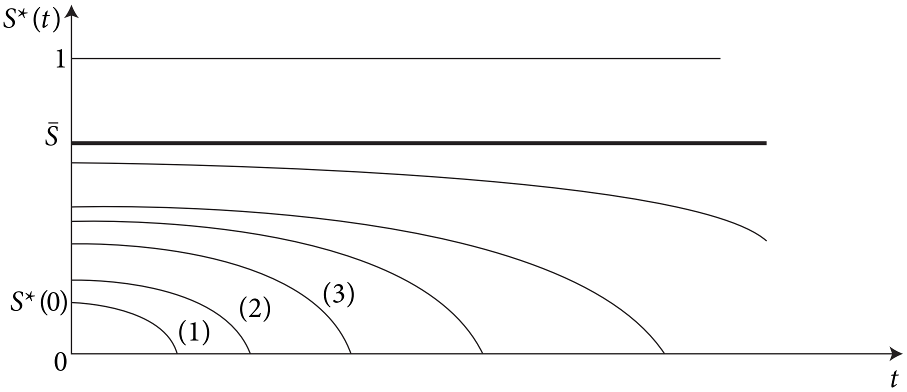

Figure 3. The optimal trajectories of investment rate (solutions of problem (9)–(10) under $t_1 \to +\infty$) and the investment turnpike $\overline{s}$

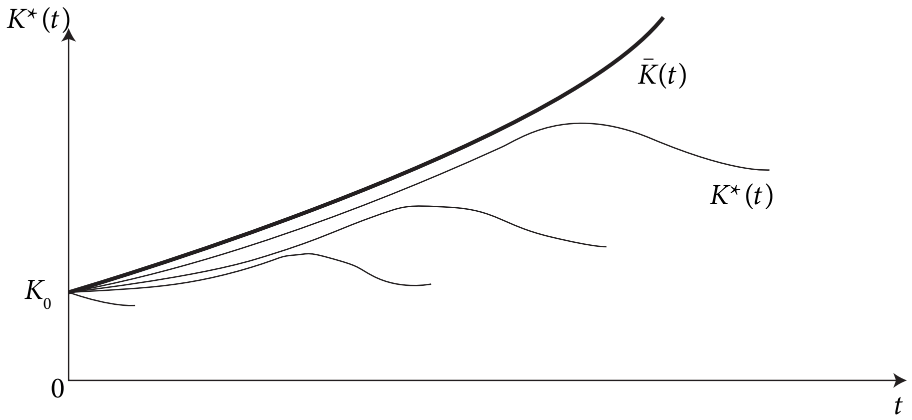

Figure 5. The optimal consumption trajectories (solutions of (9)-(10) under $t_1 \to +\infty$) and the consumption turnpike $\overline{C}(t)$

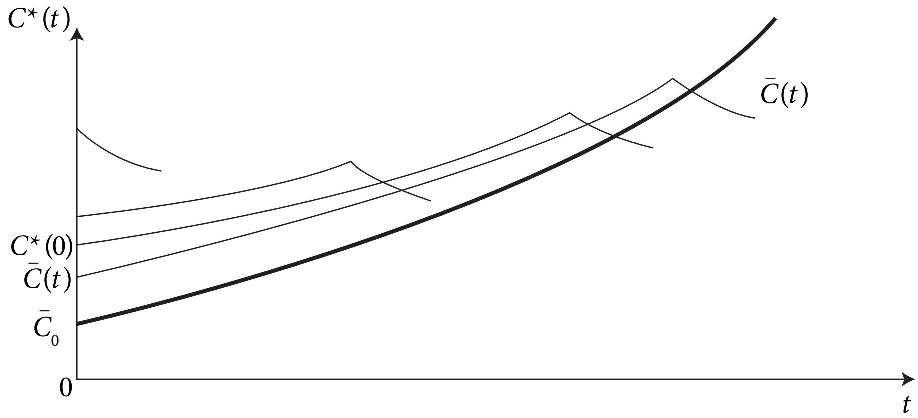

Figure 5. The optimal consumption trajectories (solutions of (9)-(10) under $t_1 \to +\infty$) and the consumption turnpike $\overline{C}(t)$

Although the turnpike growth is closer to economic reality than the optimal growth represented by problem (9)-(10), in both cases the constant capital productivity is a technological obstacle which makes a barrier to long-term growth at freely chosen speed. The turnpike growth rates of capital, production and consumption are

$$g_{\bar{K}(t)} = \frac{\dot{\bar{K}}(t)}{\bar{K}(t)} = g_{\bar{Y}(t)} = \frac{\dot{\bar{Y}}(t)}{\bar{Y}(t)} = g_{\bar{C}(t)} = \frac{\dot{\bar{C}}(t)}{C(t)} = \sigma A - \mu - \rho,$$

respectively. Taking into account that the investment efficiency rate σ may not exceed 1 *ex definitione*, we conclude that only increase of the *capital productivity A* allows for permanent growth of welfare on the grounds of the AK class models.

4. The endogenous increase of capital efficiency[8]

Economic efficiency in general, and production capital efficiency in particular, increases as time passes due to factors building human capital such as knowledge, skills, education, innovations etc. Their maintenance and development are conditional on investment. To illustrate this phenomenon we shall consider a simple model of growth, where the capital productivity increases due to investment. For simplicity we assume that investment increases the quantity of capital and (by knowledge accumulation) its productivity (quality) at the same time. So it is a version of growth model under the so-called capital-embodied technology-organizational. To simplify the exposition we assume that there is no capital depreciation ($\mu = 0$). The dynamics of capital in both models is given presently by the differential equation:

$$\dot{K}(t) = \sigma I_b(t) \tag{31}$$

or, equivalently (by equations (1)-(2))

$$\dot{K}(t) = \sigma A(t) s(t) K(t), \tag{32}$$

under initial condition (6), where as previously:

$$s \in \tilde{C}^0 \left(T \to [0,1]\right).$$

Investment[9] $I_b(t)$ increases capital according to (31) ((32) respectively) and its productivity $A(t)$ according to equation:

$$\dot{A}(t) = \delta I_b(t) \tag{33}$$

or equivalently

$$\dot{A}(t) = \delta A(t) s(t) K(t), \tag{34}$$

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[8] This section partially refers to Panek (2003), ch. 8, sect. 8.3.

[9] We preserve symbol $I_b(t)$ from section 1 for gross investment at moment t, though – when there is no capital depreciation – gross investment equals net investment ($I_b(t) = I_n(t)$).

where $\delta > 0$ is the *innovative investment efficiency rate,* $\sigma > 0$ is *investment efficiency rate*, as so far. At the initial moment $t = 0$ the capital productivity is fixed:

$$A(0) = A_0 > 0. \tag{35}$$

We shall consider first the following maximization of total consumption within horizon T:
find

$$\max \int_T A(t)\big(1-s(t)\big)K(t)dt \tag{36}$$

under constraints

$$\begin{aligned}
\dot{K}(t) &= \sigma A(t)s(t)K(t),\\
\dot{A}(t) &= \delta A(t)s(t)K(t),\\
s &\in \tilde{C}^0\big(T \to [0,1]\big),\\
K(0) &= K_0,\\
A(0) &= A_0.
\end{aligned} \tag{37}$$

From a mathematical point of view system (37) is a two-dimensional (continuous) dynamical system.

A triplet of trajectories $s(t)$, $A(t)$, $K(t)$ that satisfy the system over T is said to be a *feasible growth process*. The growth process $s^*(t)$, $a^*(t)$, $K^*(t)$ that maximizes functional (36) over the set of all feasible growth processes is called the *optimal process*.

In both control problems, considered up to this moment, the horizon was assumed to be an interval $T = [0, t_1]$, $t_1 < +\infty$ (in problem (26) we assumed $t_1 = +\infty$). Now things are different. It proves that problem (36)-(37) has the optimal solution only if the horizon is not longer than a critical value $\overline{t}$. The optimal solution is as follows[10]:

If the horizon T is short, i.e.

$$t_1 \le \theta = \frac{1}{\sigma A_0 + \delta K_0},$$

then

$$s^*(t) = 0,$$

[10] The process of solving the problem in a particular case $\sigma = 1$ is presented in Panek (2003), sect. 8.3.

$$K^*(t) = K_0, \tag{38}$$

$$A^*(t) = A_0,$$

everywhere in T. If the horizon T is long, but not longer than the critical length $\bar{t}$, i.e.

$$\theta < t_1 < \bar{t},$$

then

$$s-(t) = \begin{cases} 1 & for \quad t \in [0, \tau), \\ 0 & for \quad t \in [\tau, t_1], \end{cases} \tag{39}$$

$$K^*(t) = \begin{cases} \left[\left(\frac{1\delta}{K_0} + \frac{\delta}{\sigma A_0 - \delta K_0} \right) \exp\{-(\sigma A_0 - \delta K_0) t\} - \frac{\delta}{\sigma A_0 - \delta K_0} \right]^{-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad for \quad t \in [0, \tau), \\ K^*(\tau) \qquad\qquad\qquad\qquad\qquad for \quad t \in [\tau, t_1], \end{cases} \tag{40}$$

$$A^*(t) = \begin{cases} A_0 + \sigma^{-1}\delta\left(K^*(t) - K_0\right) & for \quad t \in [0, \tau), \\ A^*(\tau) & for \quad t \in [\tau, t_1], \end{cases} \tag{41}$$

if $\sigma A_0 \neq \delta K_0$, and

$$K^*(t) = \begin{cases} \frac{1}{(K_0)^{-1} - \delta t} & for\ t \in [0, \tau), \\ K^*(\tau) & for\ t \in [\tau, t_1], \end{cases} \tag{42}$$

if $\sigma A_0 = \delta K_0$ (the optimal trajectory of capital productivity has form (41)). The corresponding optimal consumption trajectory $C^*(t)$ is:

$$C^*(t) = \begin{cases} 0 & for\ t \in [0, \tau), \\ A^*(\tau) K^*(\tau) & for\ t \in [\tau, t_1]. \end{cases} \tag{43}$$

The moment of switching (from 1 to 0) in (39) fulfills

$$\tau = t_1 - \frac{1}{\sigma A^*(\tau) + \delta K^*(\tau)}, \tag{44}$$

with $K^*(\tau) \to +\infty$, $A^*(\tau) \to +\infty$ and $t_1 - \tau \to 0^+$ if $t_1 \to \bar{t}^-$.

The optimal capital trajectory in long term is illustrated in Fig. 6.

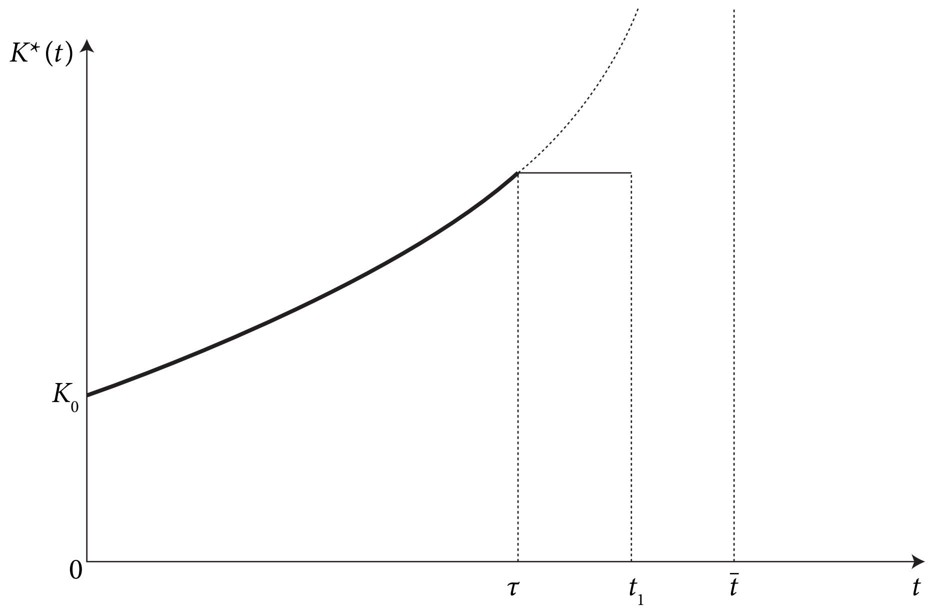

Figure 6. The optimal trajectory of capital – solution of (36)-(37) (long horizon, $T = [0, t_1]$, $t_1 < \bar{t}$)

When the horizon $T = [0, t_1]$ approaches the critical value $\bar{t}$, then the length of the first phase increases (the second one shrinks). One checks easily that in the dominating first phase

$$g_K(t) = \frac{\dot{K}^*(t)}{K^*(t)} = \sigma A^*(t) \geq \sigma A_0 \exp(\delta K_0 t) \to +\infty \quad \text{when} \quad t_1 \to \bar{t}^-,$$

$$g_A(t) = \frac{\dot{A}^*(t)}{A^*(t)} = \sigma K^*(t) \geq \sigma A_0 \exp(\delta A_0 t) \to +\infty \quad \text{when} \quad t_1 \to \bar{t}^-.$$

If feasible investment rates in problem (36)-(37) are constrained to an interval $[s_0, s_1] \subset (0, 1)$, then there exist moments $\bar{t} < +\infty$ and $\tau \in (0, t)$ such that if the horizon $T = [0, t_1]$ is short, $t_1 < \bar{t}$, then the optimal process is given by

$$s^*(t) = s_0,$$

$$K^*(t)=\left[\left(\frac{1}{K_0}+\frac{\delta}{\sigma A_0-\delta K_0}\right)\exp\{-(\sigma A_0-\delta K_0)s_0 t\}-\frac{\delta}{\sigma A_0-\delta K_0}\right]^{-1} \quad (45)$$

$$A^*(t)=A_0+\sigma^{-1}\delta\left(K^*(t)-K_0\right)$$

everywhere in T. If $\tau < t_1 < \overline{t}$, then the optimal solution of the problem is the process with optimal investment rate

$$s^*(t)=\begin{cases} s_1 & \text{for } t\in[0,\tau),\\ s_0 & \text{for } t\in[\tau,t_1],\end{cases}$$

capital

$$K^*(t)=\begin{cases}\left[\left(\frac{1}{K_0}+\frac{\delta}{\sigma A_0-\delta K_0}\right)\exp\{-(\sigma A_0-\delta K_0)s_0 t\}-\frac{\delta}{\sigma A_0-\delta K_0}\right]^{-1} & \text{for } t\in[0,\tau),\\ \left[\left(\frac{1}{K^*_{(\tau)}}-\frac{\delta}{\sigma A^*_{(\tau)}-\delta K^*_{(\tau)}}\right)\exp\{-(\sigma A^*_{(\tau)}-\delta K^*_{(\tau)}\, s_0(t-\tau)\}-\right. & \\ \left.\qquad -\frac{\delta}{\sigma A^*_{(\tau)}-\delta K^*_{(\tau)}}\right]^{-1} & \text{for } t\in[\tau,t_1)\end{cases}$$

and capital productivity $A^*(t)$ of form (45). The corresponding consumption trajectory is

$$C^*(t)=\begin{cases} A^*(t)(1-s_1)K^*(t) & \text{for } t\in[0,\tau),\\ A^*(\tau)(1-s_0)K^*(\tau) & \text{for } t\in[\tau,t_1].\end{cases}$$

As we see, within the whole horizon (in both phases) the growth rates of all basic macroeconomic variables increase unboundedly:

$$g_{K^*(t)}=\frac{\dot{K}^*(t)}{K^*(t)}\to+\infty, \text{ when } t_1\to\overline{t}^-,$$

$$g_{Y^*(t)}=\frac{\dot{Y}^*(t)}{Y^*(t)}\to+\infty, \text{ when } t_1\to\overline{t}^-.$$

And

$$g_{C^*(t)} = \frac{\dot{C}^*(t)}{C^*(t)} = \frac{\dot{A}^*(t)}{A^*(t)} + \frac{\dot{K}^*(t)}{K^*(t)} = g_{A(t)} + g_{K(t)} \to +\infty \text{ when } t_1 \to \overline{t}^{\,-}.$$

It turns out that simultaneous investment into production capital and human capital (knowledge), which in our model is embodied in technology-organizational progress (which is expressed by increasing capital productivity) leads to unbounded growth of capital, production and consumption. Moreover, this unbounded growth is (theoretically) carried out in a finite horizon ($t_1 < \overline{t}$).

Replacing consumption $C(t) = A(t)(1 - s(t))K(t)$ in (36) by its discounted utility results in the following problem:

Find:

$$\max \int_T u\big(C(t)\big)\exp(-\rho t)dt \tag{46}$$

satisfying (37).

Its solution, similarly as it was in the case of problem (36)-(37), exists only for $t_1 < \overline{t}$, where $\overline{t}$ is a finite moment in time. For $t_1 \geq \overline{t}$ the problem has no (finite) solution. If we substitute a particular form of utility function, e.g. $u(C) = \ln(C)$, then – in long horizon ($t_1 > \rho^{-1} \ln \frac{\sigma A_0 + \delta K_0}{\sigma A_0 + \delta K_0 - \rho}$ suffices) – the optimal trajectory of investment rate $s^*(t)$, the optimal trajectory of capital $K^*(t)$ and its productivity $A^*(t)$ satisfy the below given conditions:

There exists a moment $\tau \in (0, t_1)$ and numbers $\psi_1, \psi_2 > 0$, such that

$$s^*(t) = \begin{cases} 1 - \dfrac{\exp\left\{\int_0^t \left[\sigma A^*(\theta) + \delta K^*(\theta)\right] d\theta - \rho t\right\}}{(\sigma\psi_1 + \delta\psi_2)A^*(t)K^*(t)} & \textit{for } t \in [0,\ \tau), \\ 0 & \textit{for } t \in [\tau, t_1], \end{cases} \tag{47}$$

trajectories $K^*(t)$, $A^*(t)$ satisfy (37) under control (47), and at moment $t = \tau$ the following condition holds

$$(\sigma\psi_1 + \delta\psi_2)A^*(\tau)K^*(\tau) = \exp\left\{\int_0^\tau \left[\sigma A(\theta) + \delta K(\theta)\right] d\theta - \rho\tau\right\}.$$

The optimal capital trajectory is

$$K^*(t) = \begin{cases} \exp\left\{ (\sigma A_0 - \delta K_0) \int_0^t s^*(\theta) d\theta \right\} \left[K_0^{-1} - \delta \int_0^t \left\{ \exp\left[(\sigma A_0 - \delta K_0) \int_0^{\theta} s^*(\tau) d\tau \right] \times \right. \right. \\ \left. \left. \times\, s^*(\theta) d\theta \right\} \right]^{-1} & for\ t \in [0, \tau), \\ K^*(\tau) & for\ t \in [\tau, t_1]. \end{cases}$$

The optimal trajectory of capital productivity is given by (41). Shape of the optimal investment rate trajectory is similar to the one presented in Fig. 1. The optimal consumption trajectory is

$$C^*(t) = \begin{cases} \left(\sigma \psi_1 + \delta \psi_2 \right)^{-1} \exp\left\{ \int_0^t \left[\sigma A^*(\theta) + \delta K^*(\theta) \right] d\theta - \rho t \right\} & for\ t \in [0, \tau), \\ A^*(\tau) K^*(\tau) & for\ t \in [\tau, t_1]. \end{cases}$$

If $t_1 \to \bar{t}^-$, then the length of the first phase increases, the second one gets shorter, i.e. $t_1 - \tau \to 0$, when $t_1 \to \bar{t}^-$. The growth rate of capital $g_{K^*(t)} = \frac{\dot{K}^*(t)}{K^*(t)}$ and its productivity $g_{A^*(t)} = \frac{\dot{A}^*(t)}{A^*(t)}$ grow unboundedly in the first phase, when $t_1 \to \bar{t}^-$.

Analogously, the growth rate of consumption increases in this phase

$$g_{Y^*(t)} = \frac{\dot{Y}^*(t)}{Y^*(t)} = \frac{\dot{A}^*(t) K^*(t) + A^*(t) \dot{K}^*(t)}{A^*(t) K^*(t)} = g_{K^*(t)} + g_{A^*(t)} \to +\infty \text{ when } t_1 \to \bar{t}^-.$$

And the corresponding consumption growth rate

$$g_{C^*(t)} = \frac{\dot{C}^*(t)}{C^*(t)} = \sigma A^*(t) + \delta K^*(t) - \rho \to +\infty \text{ when } t_1 \to \bar{t}^-.$$

As previously, postulating the discounted utility as the criterion for growth, an infinite production and consumption growth is achieved in a finite time due to investment in capital (production and human). It is obvious that the ability to grow unboundedly is preserved in the economy when instead of consumption $C^*(t)$we use consumption p.c. $c^*(t)=\frac{C^*(t)}{L(t)}$, independent of $\frac{\dot{L}(t)}{L(t)}=\lambda$, for one has

$$g_{c^*(t)}=\frac{\dot{c}^*(t)}{\dot{c}^*(t)}=\frac{\dot{C}^*(t)L(t)-C^*(t)\dot{L}(t)}{L^2(t)}:\frac{C^*(t)}{L(t)}=\frac{\dot{C}^*(t)}{C^*(t)}-\frac{\dot{L}(t)}{L(t)}=g_{C^*(t)}-\lambda=$$

$$=\sigma A^*(t)+\delta K^*(t)-\rho-\lambda\to+\infty \text{ for } t_1\to\bar{t}^-.$$

5. Summary

Even simple mathematical models of economy as the AK-class models show the fundamental role played by human capital (here: strengthened by knowledge whose main source is scientific research) in economic growth. In these models the innovative investment causes growth of capital productivity and, in result, growth of production and consumption. This ultimate conclusion is confirmed on the grounds of other endogenous growth models, just to mention the so-called learning-by-doing, human capital or R&D models.[11]

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[11] See: Barro (2001); Becker (1962); Cichy (2009); Hall & Jones (1999); Hendricks (2002); Laitner (1993); Lucas (1988); Mannelli & Seshadri (2006); Nelson & Phelps (1966); Romer (1990); Uzawa (2005).

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